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Anisotropic step, mutual contact and area weighted festoons and parallelogram polyominoes on the triangular lattice

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Abstract

We present results for the generating functions of polygons and more general objects that can touch, constructed from two fully directed walks on the infinite triangular lattice, enumerated according to each type of step and weighted proportional to the area and the number of contacts between the directed sides of the objects. In general these directed objects are known as *festoons*, being constructed from the so-called friendly directed walks, while the subset constructed from vicious walks are staircase polygons, also known as parallelogram polyominoes. Additionally, we give explicit formulae for various first area-moment generating functions, that is when the area is summed over all configurations with a given perimeter. These results generalize and summarize nearly all known results on the square lattice, since such results can be obtained by setting one of the step weights to zero. All our results for the triangular lattice are new and hence provide the opportunity to study subtle changes in scaling between lattices. In most cases we give our results both in terms of ratios of infinite q-series and as continued fractions.

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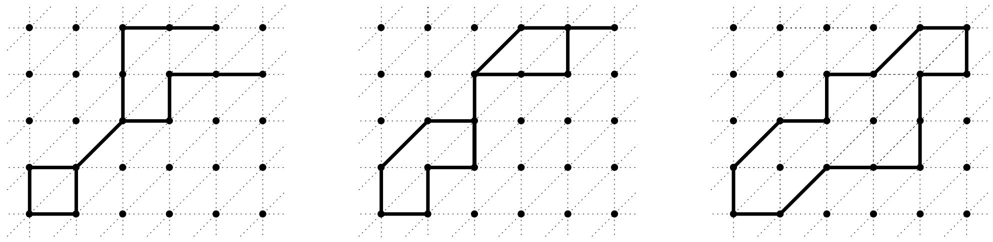
1. Introduction

Directed versions of polyominoes or polygons provide exactly solvable versions of the fundamental lattice models of lattice animals and self-avoiding polygons. They are hence a valuable testing ground for hypotheses concerning these more general models which are themselves lattice models of branched polymers and vesicles [1]. Intriguingly, recent work [2] has argued that the extent to which such directed models can mimic the behaviour of their unrestricted cousins is perhaps greater than is apparent on first glance. Staircase polygons,



(*a*) (*b*) (*c*)

Figure 1. Walk pairs on $\mathbb{T}$: (*a*) a walk pair ending apart, (*b*) a festoon, (*c*) a parallelogram polyomino.

sometimes known as parallelogram polyominoes, therefore may legitimately be used to model various ring and branched polymer systems. They have also been studied in the combinatorial literature for a long time as fundamental models that are related to other types of combinatorial objects such as lattice trees and partitions of integers [3, 4]. These directed polygons are made from two directed walks that meet at only their end-points. Papers relevant to two walks, and polygons in particular, include [5–9], with a recent review in [1]. The study of systems of two (and more) walks that may interact was popularized in [10], with the best-known result (in [11]) being for systems of non-intersecting walks. In the past much of the work on directed polygons has been completed on the square lattice, which we denote as $\mathbb{S}$. In this work we consider the triangular lattice, which we denote as $\mathbb{T}$, as it provides a future testing ground for hypotheses concerning universality of various quantities and the study of different corrections to scaling [12, 13].

In this paper we expand the usual discussion of polygons on $\mathbb{S}$ to include festoons on $\mathbb{T}$ (see below) which allows us to include weights associated with the number (and type) of contacts between the sides of the festoon: we consider different weights for site and bond double-occupation. We note in passing that the now famous problem of alternating sign matrices [14] is not only related to the six vertex model but also to osculating (touching) directed walks. Hence our results add to the literature in several ways: firstly, by considering the more general problem of festoons as well as polygons, secondly by the inclusion of contact weights in addition to area and step weights, and thirdly, and importantly, by considering the triangular lattice $\mathbb{T}$. The inclusion of area weights allows for the modelling of biological vesicles [15–17]. A reason for considering festoons interacting with contact attraction is that they provide a simple model of the process of DNA denaturation [18–22].

The types of configurations considered are shown on the $\mathbb{T}$ lattice in figure 1. Our triangular lattice is a tiling of isosceles right-angled triangles so that two adjacent triangles meeting along their hypotenuses form a square. Let us refer to those bonds as diagonal bonds since they form diagonals of squares. In this way the square lattice, rather than any arbitrary parallelogram lattice, can be obtained by the removal of a subset of bonds. This is simply an aesthetic consideration here since we consider walks with general step weights. We consider the infinite lattice with two walks sharing a common origin. Now, considering orienting the walks away from the origin, they are directed so that every edge in the walk has non-negative projection on an axis parallel to the diagonal bonds.

The classes of walk studied in this paper are subsets of pairs of walks in the infinite plane that have the same starting site, do not cross over each other, and have ending sites along a common line perpendicular to the diagonal bonds. Such general pairs of walks are here called *walk pairs*.

- A *festoon* (after [23]), is a walk pair that, in addition, has both walks ending at the same site.
- A *parallelogram polyomino* is an object made of the cells of planar lattices, the boundary of which is a festoon that has both walks meeting only at their starting and ending sites.

By expanding on the work in [24], we calculate generating functions for various classes of parallelogram polyominoes and festoons on the triangular lattice. We begin in section 2 by considering parallelogram polyominoes with anisotropic step weights only. This allows us to explore some of the combinatorial connections that these objects harbour. Next, in section 3, festoons with step weights and mutual-contact weights are discussed. We then add the consideration of weighting the area inside parallelogram polyominoes in section 4, and similarly in section 5 area is considered for festoons with step weights and mutual-contact weights. Generating functions that count the first area-moment of parallelogram polyominoes and festoons of fixed perimeters are found in the final section.

2. Parallelogram polyominoes with step weights

The boundary of a parallelogram polyomino is a walk pair from the origin, the walks of which intersect again only at their endpoints. The term *parallelogram polyomino* was introduced in [25] to describe a polyomino for which the intersection of the polyomino with every line $j = c - i$ using Cartesian coordinates (i, j), except the lines through the origin and the endpoint, are either empty or a connected segment. A similar definition was given in [5]. Parallelogram polyominoes have also been called parallelograms [6] since they can be said to have upper and lower boundaries that both climb to the right; staircase polygons [9] because their boundaries resemble staircases, and skew Ferrers diagrams [26].

With $\mathcal{PP}$ denoting the set of parallelogram polyominoes on the $\mathbb{T}$ lattice, let $\hat{P}(x, y, d)$ be a generating function

$$\hat{P}(x, y, d) = \sum_{p\in\mathcal{PP}} x^{a(p)} y^{b(p)} d^{c(p)} \tag{2.1}$$

that enumerates parallelogram polyominoes on the $\mathbb{T}$ lattice by types of edges in their boundaries. A polyomino p with $a(p)$ horizontal, $b(p)$ vertical and $c(p)$ diagonal edges in total in its boundary contributes an $x^{a(p)} y^{b(p)} d^{c(p)}$ term to $\hat{P}(x, y, d)$. Because a polyomino is defined as a union of cells, the single site is not defined as a (zero-area) polyomino. Thus $\hat{P}(x, y, d) = 2xyd + x^2y^2 + x^2d^2 + y^2d^2 + 7x^2y^2d^2 \ldots.$

Several methods have been used to find $\hat{P}(x, y, 0)$, the $\mathbb{S}$ lattice case of $\hat{P}(x, y, d)$, since it is useful as a test case before embarking on a more complex problem. Summaries of some of these methods are given in [23, 1]. A further method that is described in (for example) [27] is a convolution of two walk pairs that both end a given distance apart.

Many methods for deriving $\hat{P}(x, y, 0)$ can be extended to the general $\mathbb{T}$ lattice case. In this section, a first expression for $\hat{P}(x, y, d)$ is derived using a method due to Temperley from [28] that has been revived for solving lattice enumeration problems [29]. A simpler expression for $\hat{P}(x, y, d)$ is then deduced from a different derivation that also uses this method.

2.1. Deriving $\hat{P}(x, y, d)$ via left height of first column

A parallelogram polyomino can be considered as a set of columns, each one cell wide, where each column has at least one vertical edge in common with the columns on either side of it, with two additional conditions. Firstly, the columns to the right of any column have no cells

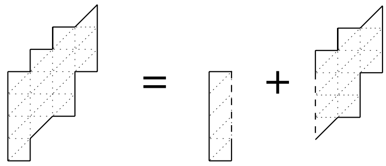


Figure 2. The decomposition of a four-column parallelogram polyomino into its first column and remainder; the dashed edges are internal edges of the polyomino.

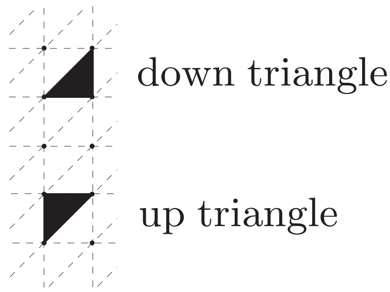


Figure 3. An up triangle and a down triangle on the $\mathbb{T}$ lattice.

below the rightmost point of the bottommost cell of the column and, secondly, any particular column has no cells higher than the leftmost points of the topmost cells in any column to the right. These conditions ensure overlap between successive columns and the directness of the boundary walks. For example, the polyomino on the left side of figure 2 has four columns.

For later reference we note that each column, and so each parallelogram polyomino, is a union of triangular cells (tiles or faces) of two types. We will denote these types as *up triangles* for cells whose apex points northwest and *down triangles* for cells whose apex points southeast (see figure 3).

A natural decomposition of the cells of a parallelogram polyomino, shown in figure 2, is to separate the first column, and so split the polyomino into a single column (henceforth the *first column*) and another, possibly empty, parallelogram polyomino (the *remainder*). The perimeter of the overall polyomino is then the sum of the perimeters of the first column and the remainder less the lengths of the vertical edges, originally internal to the polyomino, that the split has made external. Decomposition of a parallelogram polyomino by rows is equivalent to decomposition by columns and is not considered here.

The set of parallelogram polyominoes is here classified by the number of vertical edges, n, along the left side of the first column of a polyomino. This characteristic is here called the *left height* of the polyomino. If $S_n(x, y, d)$ (or S_n for short) is a generating function enumerating parallelogram polyominoes of left height n by types of edges in their boundaries, then

$$\hat{P}(x, y, d) = \sum_{n\geqslant 0} S_n. \tag{2.2}$$

The function $\hat{P}(x, y, d)$ is found below by deriving recurrences for S_n. The smallest possible left height of a parallelogram polyomino on the $\mathbb{T}$ lattice is zero. In this case the first column can be only a single down triangle. As a self-standing polyomino (with empty remainder) this first column contributes an xyd term to S_0. A single down triangle has

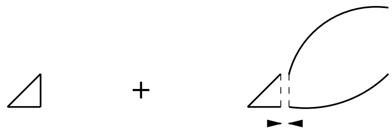


Figure 4. Parallelogram polyominoes with left height of zero.

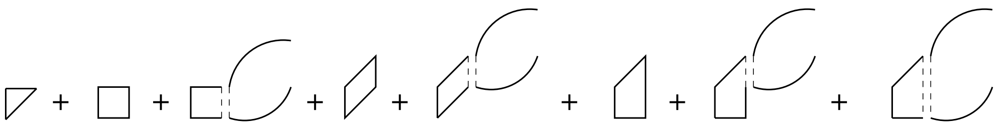


Figure 5. Parallelogram polyominoes with left height of one.

right height of one, so, if the remainder of the polyomino is non-empty, the requirement for overlap between columns means that the remainder must have left height of at least one. The possibilities are shown schematically in figure 4; adding the two cases gives

$$S_0 = xyd\left(1 + y^{-2}\sum_{j\geqslant 1} S_j\right) = xyd + xy^{-1}d\sum_{j\geqslant 1} S_j \tag{2.3}$$

where in the latter case allowance has been made for gluing the first column back on to the remainder and so reducing the number of vertical edges back to the number in the overall polyomino.

When the left height of a polyomino is one, there are four choices for the first column. These are shown in figure 5 along with possible remainders for each choice of first column. It can be seen from this figure that the function S_1 must then satisfy a relation to S_n:

$$\begin{aligned} S_1 &= xyd + (x^2y^2 + d^2y^2)\left(1 + y^{-2}\sum_{j\geqslant 1} S_j\right) + xy^3d\left(1 + y^{-2}\sum_{j\geqslant 1} S_j + y^{-4}\sum_{j\geqslant 2} S_j\right) \\ &= (x^2 + d^2 + xyd)S_1 + (x^2 + d^2 + xyd + xy^{-1}d)\left(y^2 + \sum_{j\geqslant 2} S_j\right). \end{aligned} \tag{2.4}$$

If the same procedure is followed for the set of parallelogram polyominoes of left height of at least 2, the general relation

$$S_{n+1} = y(y + xd)S_n + (x^2 + d^2 + xyd)S_{n+1} + (x^2 + d^2 + xyd + xy^{-1}d)\sum_{j\geqslant n+2} S_j \tag{2.5}$$

is obtained. This relation can be manipulated into the recurrence

$$(y + xd)S_{n+2} - (1 + y^2 - x^2 - d^2)yS_{n+1} + y^2(y + xd)S_n = 0 \qquad n \geqslant 1 \tag{2.6}$$

which is a second-order difference equation with constant coefficients, and so has solutions of the form

$$S_n = A\lambda_+^n + B\lambda_-^n \tag{2.7}$$

where A and B are constants (independent of n) to be determined, and $\lambda_\pm(x, y, d)$ are the roots of the quadratic

$$(y + xd)\lambda^2 - (1 + y^2 - x^2 - d^2)y\lambda + y^2(y + xd) = 0 \tag{2.8}$$

i.e.,

$$\lambda_\pm(x, y, d) = y\frac{1 + y^2 - x^2 - d^2 \pm \sqrt{(1 + y^2 - x^2 - d^2)^2 - 4(y + xd)^2}}{2(y + xd)}. \tag{2.9}$$

A generating function enumerating parallelogram polyominoes of left height n by the number of edges in their boundaries via the variable z is the isotropic perimeter function $S_n(z, z, z)$. Since a parallelogram polyomino of left height n must have at least n vertical edges in its boundary, by simply expanding $\lambda_\pm$ it can be readily seen that A must be zero and only λ_- contributes to S_n. The constant B can be calculated from equations (2.7) and (2.4) as

$$B = \frac{(x + dy)(d + xy)}{y + xd}. \tag{2.10}$$

A perimeter generating function for parallelogram polyominoes is the sum of S_n for all values n of the left height, so, from equation (2.2),

$$\begin{aligned}\hat{P}(x, y, d) &= \sum_{n\geqslant 0} S_n = S_0 + \frac{(x + yd)(d + xy)}{y + xd}\sum_{n\geqslant 1}\lambda_-^n \\ &= xyd + \frac{(x + yd)(d + xy)}{y}\sum_{n\geqslant 1}\lambda_-^n.\end{aligned} \tag{2.11}$$

Whilst this expression of $\hat{P}(x, y, d)$ can be simplified by substituting in the expression of equation (2.9) for λ_-, the (formal) geometric series factor $\sum_{n\geqslant 1}\lambda_-^n$ suggests that there could be a derivation of the perimeter generating function in which objects of size n are enumerated by λ_-^n. Part of such a derivation is given in the next subsection, and is shown to lead to a simpler expression for $\hat{P}(x, y, d)$.

2.2. *Deriving $\hat{P}(x, y, d)$ via right height of first column*

Another way to enumerate parallelogram polyominoes by the types of steps in their boundary via the method of Temperley introduced in the last subsection is to take the right (rather than left) height of the first column to be the important characteristic of the polyomino. Let the *right height* of a polyomino be the number of vertical edges on the right side of the first column, and let $D_n(x, y, d)$ be a generating function enumerating parallelogram polyominoes of right height n by types of edges in their boundaries, so that

$$\hat{P}(x, y, d) = \sum_{n\geqslant 0} D_n(x, y, d). \tag{2.12}$$

When the right height of a polyomino is zero, its first column can be only the single up triangle. This is the only possible polyomino contributing to $D_0(x, y, d)$ since the lack of edges on the right-hand side of the column does not allow multi-column polyominoes of zero right height. Thus

$$D_0(x, y, d) = xyd. \tag{2.13}$$

If the right height of a polyomino is nonzero, then there are four possible first columns, depending on whether the top and bottom edges of the column are horizontal or diagonal. These four possibilities are shown in the left side of figure 6. The four polyominoes that have

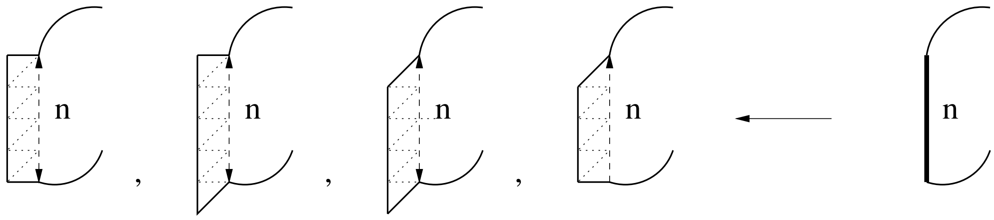


Figure 6. Parallelogram polyominoes with right height of n can be constructed from a lattice object that has compressed the first column as the edges in the right height of the column.

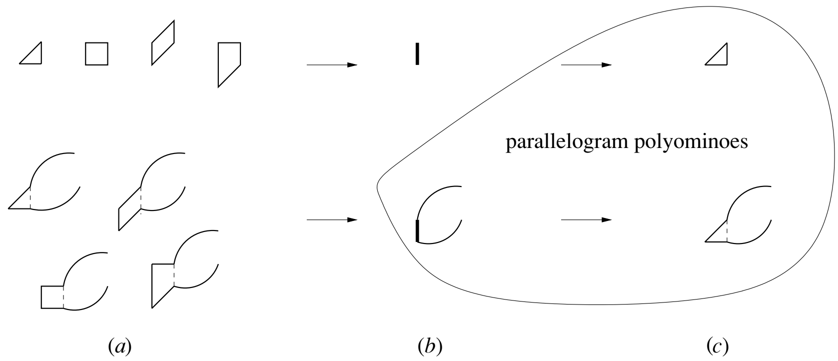


Figure 7. Obtaining an expression for $\hat{P}(x, y, d)$: (a) Parallelogram polyominoes with right height of 1; (b) the set of lattice objects with perimeter generating function $T_1(x, y, d)$; (c) parallelogram polyominoes with left height of 0; the ringed objects are then all the parallelogram polyominoes.

a common remainder and the same sites of overlap between first column and remainder can all be constructed from a single object that is the remainder with a line of length n overlapping its first column. If $T_n(x, y, d)$ is introduced as a generating function enumerating the resulting 'remainder with line' lattice objects by their perimeter, then

$$D_n(x, y, d) = (x^2 + d^2 + xyd + xy^{-1}d)T_n(x, y, d). \tag{2.14}$$

If equations (2.13) and (2.14) are substituted in equation (2.12), comparison of that equation with equation (2.11) shows that $T_n(x, y, d) = \lambda_-^n$. A more direct combinatorial derivation of this last result, i.e., one that did not rely on the results in section 2.1, would reduce the length of the derivation of $\hat{P}(x, y, d)$ and so would be interesting to see.

Nonetheless, these extra functions $T_n(x, y, d)$ are useful for finding a simple expression for $\hat{P}(x, y, d)$: the set of objects enumerated by $T_1(x, y, d)$ can be modified to give the set of parallelogram polyominoes. Indeed, the objects enumerated by $T_1(x, y, d)$ are formed from the parallelogram polyominoes of right height 1 by compressing the first column of the polyomino into the single vertical edge on its right side (as shown in figures 7(a) and (b)). The remainder of a polyomino of right height 1 is either a parallelogram polyomino itself or is empty. Thus $T_1(x, y, d) - y^2 = \lambda_- - y^2$ is also a perimeter generating function for parallelogram polyominoes of left height at least 1.

As shown in figure 7(c), the set of parallelogram polyominoes of zero left height can be formed by adding a column of a single down triangle to the left of each object enumerated by

$T_1(x, y, d)$, so

$$\hat{P}(x, y, d) = \lambda_- - y^2 + xy^{-1}d\lambda_-$$
$$= \frac{1 - x^2 - y^2 - d^2 - \sqrt{(1 - x^2 + y^2 - d^2)^2 - 4(y + xd)^2}}{2} \tag{2.15}$$

which can be rewritten as

$$\hat{P}(x, y, d) = \frac{1 - x^2 - y^2 - d^2 - \sqrt{(1 - x^2 - y^2 - d^2)^2 - 4(x^2y^2 + x^2d^2 + y^2d^2 + 2xyd)}}{2}. \tag{2.16}$$

2.3. Special cases

Case 2.1. On the isotropic $\mathbb{S}$ lattice we have

$$\hat{P}(z, z, 0) = \frac{1 - 2z^2 - \sqrt{1 - 4z^2}}{2} = z^2(C(z^2) - 1) \tag{2.17}$$

so the number of parallelogram polyominoes of perimeter $2n + 2$ edges on the $\mathbb{S}$ lattice is the nth Catalan number c_n [30, 5]. This result, and also the corresponding anisotropic function $\hat{P}(x, y, 0)$, are both well-known.

Case 2.2. Another interesting case combinatorially is when one lets $x = y = z$ and $d = z^2$:

$$\hat{P}(z, z, z^2) = \frac{1 - 2z^2 - z^4 - \sqrt{(1 - 6z^2 + z^4)(1 + z^2)^2}}{2} = z^2(z^2R(z^2) + R(z^2) - 1) \tag{2.18}$$

so the number of parallelogram polyominoes of perimeter $2n + 2$ on the $\mathbb{T}$ lattice where diagonal steps are treated as twice the length of vertical or horizontal steps is $r_n + r_{n-1}$, where r_n is the nth large Schröder number [31, 32, 4].

Case 2.3. On the isotropic triangular lattice the perimeter only generating function for parallelogram polyominoes is

$$\hat{P}(z, z, z) = \frac{1 - 3z^2 - \sqrt{(1 - 2z - 3z^2)(1 + z)^2}}{2} = z^2\left(zM(z) + M(z) - 1\right) \tag{2.19}$$

so the number of parallelogram polyominoes of perimeter $n + 1$ on the $\mathbb{T}$ lattice is $m_n + m_{n-1}$, where m_n is the nth Motzkin number [33–37].

Case 2.4 (Enumerating by endpoint). Let $\hat{P}(x, y, d; \nu, \rho)$ be a generating function enumerating parallelogram polyominoes by types of edges in their boundaries and also by the numbers of columns (with ν being conjugate variable) and rows (with ρ being conjugate variable). Because of the directedness, the numbers of columns and rows are equivalent to the coordinates of the endpoint of the two directed walks making up the boundary of the polyomino as described in the introduction. An expression for this function can be found by making the substitutions $x \to \sqrt{\nu}x$, $y \to \sqrt{\rho}y$ and $d \to \sqrt{\nu\rho}d$ in equation (2.16), so that

$$\hat{P}(x, y, d; \nu, \rho) = \frac{U - \sqrt{U^2 - 4\nu\rho(x^2y^2 + \nu x^2d^2 + \rho y^2d^2 + 2xyd)}}{2} \tag{2.20}$$

where

$$U = 1 - \nu x^2 - \rho y^2 - \nu\rho d^2. \tag{2.21}$$

For example, on the $\mathbb{S}$ lattice

$$\hat{P}(z, z, 0; \nu, \rho) = \frac{1 - (\nu + \rho)z^2 - \sqrt{1 - 2(\nu + \rho)z^2 + (\nu - \rho)^2z^4}}{2}. \tag{2.22}$$

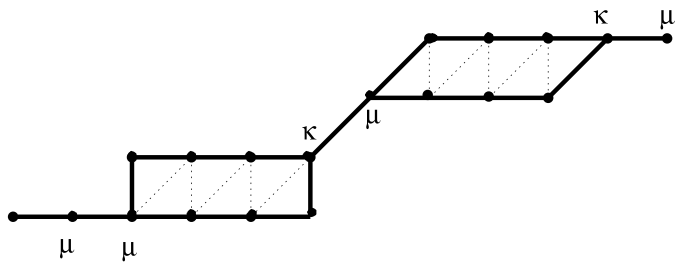


Figure 8. Weights of contact representing interaction between the walks of a walk pair.

The number of (configurations of) parallelogram polyominoes on the $\mathbb{S}$ lattice that have endpoint at $(r, n-r)$, where we use a standard Cartesian coordinate system and $n, r \geqslant 1$, is the coefficient $[\nu^r \rho^{n-r} z^n]$ of $\hat{P}(z, z, 0; \nu, \rho)$, which is [30]

$$N(r, n-r) = \frac{1}{n-1}\binom{n-1}{r}\binom{n-1}{r-1} \tag{2.23}$$

where $N(r, n-r)$ is a Narayana number[1].

Further to this, a parallelogram polyomino in which each column can be coloured without restriction any of k colours is often called k-coloured. Coloured polyominoes have been considered on the $\mathbb{S}$ lattice in [42–45, 4], particularly two-coloured parallelogram polyominoes (*zebras*). Enumeration of k-coloured polyominoes can be achieved by weighting each column via $\nu = k$. Thus, for example, an isotropic perimeter generating function for k-coloured parallelogram polyominoes on the $\mathbb{S}$ lattice is

$$\hat{P}(z, z, 0; k, 1) = \frac{1 - (k+1)z^2 - \sqrt{1 - 2(k+1)z^2 + (k-1)^2 z^4}}{2}. \tag{2.24}$$

3. Festoons with contact weights

A walk pair can be decomposed into components in which the walks are apart, i.e. parallelogram polyominoes, and sections in which the walks are together on shared edges. A *contact* between the two walks of a walk pair is defined to be a site shared by both walks, and a *return contact* a contact other than the initial contact at the origin. If the walks take different edges to arrive at a site (and so are the final edges of each walk of a parallelogram polyomino), the contact at the site is here weighted by κ; if both walks use the same edge to arrive at a site, this contact is here weighted by μ (see figure 8).

The term *festoon* was introduced in [23] to refer to the signed festoons that are described in case 3.2 below. Here, the term festoon is used to denote similar lattice objects: walk pairs in which the walks of the pair start and end at the same sites. These general festoons appear (unnamed) in an exercise in [4], and also are essentially the same as the ∞-friendly walkers introduced in [46]. The zero-edge walk pair, i.e., the site at the origin, is classed as a festoon. Since all other festoons are concatenations of parallelogram polyominoes, with generating function $\kappa\hat{P}(x, y, d)$, and sections in which the walks of the walk pair occupy the same edges

[1] The numbers $N(r, n-r)$ were for a time called Runyon numbers [38, p 17], and also in [7, 39], but now are known as Narayana numbers. They were introduced in [40]. They are a generalization of Catalan and Schröder numbers, and also are used to describe return walks, for which see [41, 32]. For more detailed descriptions of Narayana numbers, see the references in [4].

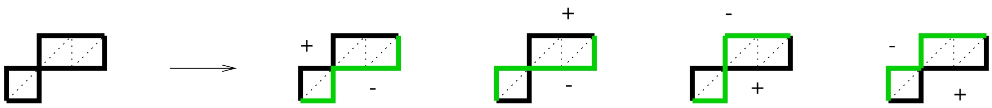


Figure 9. Signed festoons generated from a single festoon.

of the lattice, with generating function $\mu(x^2 + y^2 + d^2)$, then $\mathbf{P}(x, y, d; \kappa, \mu)$, the generating function enumerating contact-weighted festoons by types of edges, is

$$\mathbf{P}(x, y, d; \kappa, \mu) = \frac{1}{1 - \mu(x^2 + y^2 + d^2) - \kappa \hat{P}(x, y, d)}. \tag{3.25}$$

Festoons weighted by contacts provide a simple model of DNA denaturation [18–22].

3.1. Special cases

Case 3.1 (Without weights). If κ and μ are both unity, then the contact weights are not present. Thus a generating function enumerating festoons by types of edges alone (without contact weights) is, after quite a bit of rearranging

$$P(x, y, d) = \mathbf{P}(x, y, d; 1, 1) = \hat{P}(x, y, d)\frac{1}{x^2y^2 + x^2d^2 + y^2d^2 + 2xyd}. \tag{3.26}$$

The $\mathbb{S}$ lattice case of this expression, i.e. $P(x, y, 0)$, is essentially derived in [46], where similar objects called ∞-friendly walks are considered, and an expression for $P(x, y, 0)$ is also implicit in [7, 23]. Finally, the isotropic perimeter function on the $\mathbb{S}$ lattice, $P(z, z, 0)$, is equal to $C(z^2)$, where $C(z)$ is the generating function of the Catalan numbers.

Case 3.2 (Signed festoons). A *signed festoon* is defined here to be a festoon in which the walks of the walk pair may cross over. The walk that arrives at a return contact site as the upper walk need not leave as the upper walk, and similarly for the lower walk. Instead, a signed festoon has a positive walk and a negative walk, so if there are n return contacts between the walks, an unsigned festoon generates 2^n signed festoons. For example, four signed festoons can be formed from the festoon on the left of figure 9.

In a signed festoon, there are two possibilities for each parallelogram polyomino component: one with the initial upper walk (from the origin) as the upper walk, and one with it as the lower walk. A generating function enumerating signed festoons by types of edges is then

$$\mathbf{P}(x, y, d; 2, 1) = \frac{1}{\sqrt{(1 - x^2 - y^2 - d^2)^2 - 4(x^2y^2 + x^2d^2 + y^2d^2 + 2xyd)}}. \tag{3.27}$$

Signed festoons on the $\mathbb{S}$ lattice have been used in proofs in [7, 23]. On this lattice

$$\begin{aligned}\mathbf{P}(x, y, 0; 2, 1) &= \frac{1}{\sqrt{1 - 2x^2 - 2y^2 - 2x^2y^2 + x^4 + y^4}} \\ &= \sum_{n\geqslant 0}\sum_{r\geqslant 0}^{n} \binom{n}{r}^2 x^r y^{n-r}.\end{aligned} \tag{3.28}$$

The second equality above stems from the argument that on the $\mathbb{S}$ lattice there are $\binom{n}{r}$ single walks, so $\binom{n}{r}^2$ signed festoons, that start at the origin and have end point at $(r, n-r)$. There are then $\binom{2n}{r}$ signed festoons on the $\mathbb{S}$ lattice with perimeter of $2n$ edges. Signed festoons on triangular lattices however do not appear to have been studied previously.

Case 3.3 (Osculating walk pairs). If $\kappa = 1$ and $\mu = 0$, then the walks in a festoon are allowed to meet at sites but not to occupy the same edge. The walk pair of such a festoon is said to be *osculating* [47]. Osculating walk pair functions are here denoted by the symbol $\overset{\vee}{\wedge}$. Thus

$$\overset{\vee}{\hat{P}}(x, y, d) = \mathbf{P}(x, y, d; 1, 0) = \frac{1}{1 - \hat{P}(x, y, d)}. \tag{3.29}$$

The $\mathbb{S}$ lattice case of this is essentially the generating function given in [46]; further to that result, the isotropic perimeter generating function for osculating walk pairs on the $\mathbb{S}$ lattice is

$$\overset{\vee}{\hat{P}}(z, z, 0) = \frac{2}{1 + 2z^2 + \sqrt{1 - 4z^2}} = z^{-2}F(z^2) \tag{3.30}$$

where $F(z)$ is the generating function of the Fine numbers [48].

4. Parallelogram polyominoes enumerated by steps and area

4.1. Standard area

With $\mathcal{PP}$ again denoting the set of parallelogram polyominoes on the $\mathbb{T}$ lattice, let $\hat{Q}(x, y, d; q)$ be the generating function

$$\hat{Q}(x, y, d; q) = \sum_{p \in \mathcal{PP}} x^{a(p)} y^{b(p)} d^{c(p)} q^{i(p)} \tag{4.31}$$

that enumerates parallelogram polyominoes by types of edges in their boundaries and standard area. A parallelogram polyomino of $i(p)$ triangular cells with $a(p)$ horizontal, $b(p)$ vertical and $c(p)$ diagonal edges in total in its boundary contributes an $x^{a(p)}y^{b(p)}d^{c(p)}q^{i(p)}$ term to $\hat{Q}(x, y, d; q)$.

In section 2, the polyominoes in the set $\mathcal{PP}$ were decomposed into their first column and a (possibly empty) remainder, from which the perimeter generating function $\hat{P}(x, y, d)$ was found by summing the perimeters of the two components of each polyomino. In this section, the complementary approach of adding a column to a (possibly empty) polyomino in order to count the area and perimeter of the augmented polyomino is used. The derivation presented here is based upon that used to derive $\mathbb{S}$ lattice functions, for example in [49, 50]; it is a functional version of the method used by Temperley [28]. Here $\hat{Q}(x, y, d; q)$ is found as a special case of a function that, in addition to perimeter and area, also enumerates polyominoes by their left height. Let the variable s count the left height of a parallelogram polyomino (defined in section 2), and let $\hat{X}(s; x, y, d; q)$ (or $\hat{X}(s)$ for short) be the generating function

$$\hat{X}(s) = \sum_{p \in \mathcal{PP}} s^{\mathrm{left}(p)} x^{a(p)} y^{b(p)} d^{c(p)} q^{i(p)} \tag{4.32}$$

that enumerates parallelogram polyominoes by left height, types of edges in their boundaries and standard area, where left(p) is the left height of the parallelogram polyomino p. The desired perimeter-area generating function, $\hat{Q}(x, y, d; q)$, is then the function $\hat{X}(1)$.

It is convenient to derive $\hat{X}(s)$ as the sum of two functions: one function, here denoted $\hat{X}_1(s)$, enumerating single-column polyominoes, and the other, $\hat{X}_2(s)$, enumerating multi-column polyominoes.

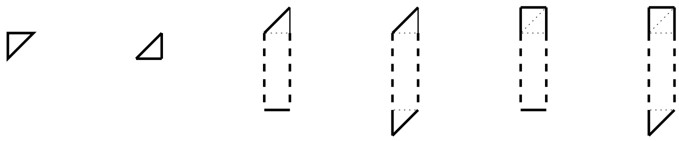


Figure 10. Types of possible single-column parallelogram polyominoes.

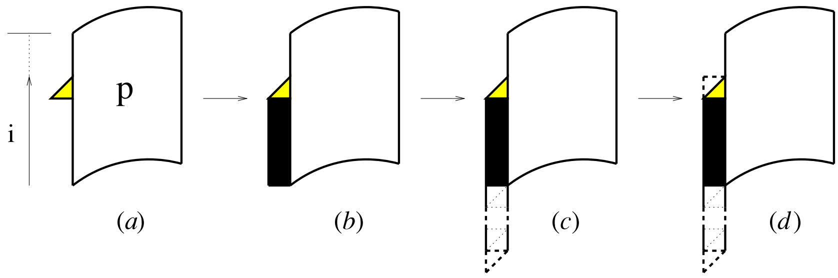


Figure 11. Addition of a column to a parallelogram polyomino p with the first down triangle of the column in grey and other required cells in black.

The function $\hat{X}_1(s)$ can be found by building the single column around its uppermost down triangle. If the column has no down triangles, then it must consist of a single up triangle. Otherwise, below the uppermost down triangle in the column there is a (possibly empty) sequence of up–down triangle pairs ('squares') which may in addition have an up triangle at its base, and above the uppermost down triangle there is either no further cell or a single up triangle. These possibilities are shown in figure 10.

Consideration of the perimeters, areas and left heights of these single columns leads to the following equation:

$$\begin{aligned}\hat{X}_1(s) &= sxydq + \frac{xydq}{1-sy^2q^2} + \frac{sy^2d^2q^2}{1-sy^2q^2} + \frac{sx^2y^2q^2}{1-sy^2q^2} + \frac{s^2xy^3dq^3}{1-sy^2q^2} \\ &= sxydq + \frac{y(sxyq^2+dq)(x+sydq)}{1-sy^2q^2}.\end{aligned} \tag{4.33}$$

The construction of a single column in finding $\hat{X}_1(s)$ is unnecessarily complicated; it has been introduced because a similar construction is now used to find an expression for $\hat{X}_2(s)$. For a parallelogram polyomino p of non-zero left height, an augmented polyomino that has an additional column on the left can be formed by attaching a down triangle to one of the vertical edges counted in the left height of p, and constructing a column around this down triangle as described below: Suppose the down triangle is attached to the ith vertical edge in the left height of p, where $1 \leqslant i \leqslant \text{left}(p)$, as in figure 11($a$). Then

- For the augmented polyomino to be a parallelogram polyomino, the $i-1$ vertical edges in the left height of p below the edge to which the down triangle has already been attached must have a up–down triangle pair ('square') to their left in this additional column, as in figure 11(b).
- At heights below the base of p, the additional column can have an arbitrary number of (alternately) up and down triangles, and have as its base cell either an up or down triangle, as in figure 11(c)
- An up triangle can also be added above the first-attached down triangle, as in figure 11(d).

Consideration of how the perimeter, area and left height of p change by augmenting it with this additional column leads to the following equation:

$$\hat{X}_2(s) = (sxq^2 + y^{-1}dq)\frac{(x + sydq)}{1 - sy^2q^2}\sum_{i=1}^{\mathrm{left}(p)}(sq^2)^i \hat{X}(s)$$
$$= \frac{(sxyq^2 + dq)(x + sydq)}{y(1 - sy^2q^2)(1 - sq^2)}(\hat{X}(1) - \hat{X}(sq^2)) \tag{4.34}$$

where in the latter equality the terms from polyominoes of zero left height cancel inside the last set of brackets. Thus

$$\hat{X}(s) = \hat{X}_1(s) + \hat{X}_2(s)$$
$$= sxydq + \frac{q(sxyq + d)(x + sydq)}{1 - sy^2q^2}\left(y + \frac{\hat{X}(1) - \hat{X}(sq^2)}{y(1 - sq^2)}\right). \tag{4.35}$$

An iterative technique for functional equations [51, 24] can then be employed at this point to obtain the desired $\hat{Q}(x, y, d; q)$, i.e., $\hat{X}(1)$. Basically the idea is that equation (4.35) provides an equation expressing $\hat{X}(s)$ in terms of $\hat{X}(1)$ and $\hat{X}(sq^2)$ so that by substituting $s = sq^2$ into equation (4.35) one obtains an equation giving $\hat{X}(sq^2)$ in terms of $\hat{X}(1)$ and $\hat{X}(sq^4)$ and so one can rewrite equation (4.35) as an equation giving $\hat{X}(s)$ in terms of $\hat{X}(1)$ and $\hat{X}(sq^4)$. By an indefinite series of substitutions, a convergence argument and a final substitution of $s = 1$, one can obtain a functional equation for $\hat{X}(1)$. Hence

$$\hat{Q}(x, y, d; q) = qy\frac{\sum_{n\geqslant 0}(xd(1 + q^{2n}) + (x^2 + d^2)yq^{2n+1})(1 - y^2q^{2n+2})^{-1}V_n(x, y, d; q)}{\sum_{n\geqslant 0} V_n(x, y, d; q)} \tag{4.36}$$

where

$$V_n(x, y, d; q) = \frac{(-1)^n q^n}{y^n(y^2q^2; q^2)_n(q^2; q^2)_n}\prod_{j=1}^{n}(xyq^{2j-1} + d)(x + ydq^{2j-1}). \tag{4.37}$$

This perimeter-area generating function for parallelogram polyominoes on the $\mathbb{T}$ lattice is new; the $\mathbb{S}$ lattice case, $\hat{Q}(x, y, 0; q)$, is well-known.

Case 4.1 (Parallelogram polyominoes on the $\mathbb{S}$ lattice). From equations (4.36) and (4.37) a generating function enumerating parallelogram polyominoes on the $\mathbb{S}$ lattice by anisotropic perimeter and area can be derived as

$$\hat{Q}(x, y, 0; q) = y^2\frac{\sum_{n\geqslant 0}(-1)^n x^{2n+2}q^{(n+1)(n+2)}(y^2q^2; q^2)_{n+1}^{-1}(q^2; q^2)_n^{-1}}{\sum_{n\geqslant 0}(-1)^n x^{2n}q^{n(n+1)}(y^2q^2; q^2)_n^{-1}(q^2; q^2)_n^{-1}}. \tag{4.38}$$

On the $\mathbb{S}$ lattice, a parallelogram polyomino is a collection of square cells. There are no diagonal edges on this lattice, so both the upper and lower walk boundary of the polyomino must contain the same number of horizontal (and vertical) edges. Many previous expressions for a perimeter-area generating function for parallelogram polyominoes on the $\mathbb{S}$ lattice therefore enumerate edges in the semi-perimeter and count area as the number of square cells, and so derive $\hat{Q}(\sqrt{x}, \sqrt{y}, 0; \sqrt{q})$.

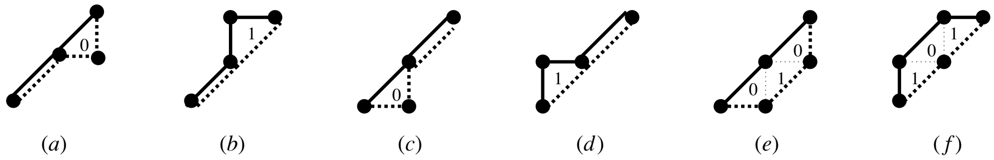


Figure 12. The $\triangle$-area of each festoon that has anisotropic perimeter term of xyd^3, as the sum of the number of up triangles.

4.2. $\triangle$-area

The $\triangle$-area of a parallelogram polyomino or festoon on the $\mathbb{T}$ lattice is the number of up triangles enclosed between the walks of the parallelogram polyomino or festoon, respectively.

Example 4.1. Of the six festoons on the $\mathbb{T}$ lattice that have anisotropic perimeter term of xyd^3, the walks of the festoons (*a*) and (*c*) in figure 12 enclose no $\triangle$-area, the walks of (*b*), (*d*) and (*e*) enclose one unit and (*f*) encloses two units.

In this subsection we restrict our discussion to parallelogram polyominoes. In order to derive $\triangle$-area functions, however, it is convenient in this section to consider the complement of the $\triangle$-area, i.e., the number of down triangles of area enclosed by the walks of the parallelogram polyomino, which is the $\triangledown$-area. Here, generating functions derived using the number of down triangles enclosed as the area are given a down-pointing triangular subscript ($_\triangledown$). An expression for $\hat{Q}_\triangledown(x, y, d; q)$, the generating function enumerating parallelogram polyominoes by their $\triangledown$-area and edge types can be obtained from the process used in the last subsection for the standard area function, and is

$$\hat{Q}_\triangledown(x, y, d; q) = y\frac{\sum_{n\geqslant 0}(xd(q+q^n) + (x^2+d^2)yq^{n+1})(1-y^2q^{n+1})^{-1}V_{\triangledown,n}(x, y, d; q)}{\sum_{n\geqslant 0} V_{\triangledown,n}(x, y, d; q)} \tag{4.39}$$

where

$$V_{\triangledown,n}(x, y, d; q) = \frac{(-q)^n}{y^n(y^2q; q)_n(q; q)_n}\prod_{j=1}^{n}(xyq^{j-1}+d)(x+ydq^{j-1}). \tag{4.40}$$

Reflection in the line $j = i$ maps the set of parallelogram polyominoes to itself, but switches the down and up triangles of each polyomino. The $\triangle$-area function, $\hat{Q}_\triangle(x, y, d; q)$, thus can be found from this reflection as

$$\hat{Q}_\triangle(x, y, d; q) = \hat{Q}_\triangledown(y, x, d; q) \tag{4.41}$$

but apparently not by direct application of the process of the previous subsection. Note that there is no apparent simple relationship between $\hat{Q}_\triangledown(x, y, d; q)$ and $\hat{Q}(x, y, d; q)$.

5. Festoons with step, contact weights and area

Now that functions enumerating parallelogram polyominoes by perimeter and area have been obtained, the corresponding functions for festoons can be deduced. Since a festoon is a concatenation of parallelogram polyominoes and sections in which the two walks are together, then a generating function enumerating contact-weighted festoons on the $\mathbb{T}$ lattice by types of edges (anisotropic perimeter) and standard area is

$$\mathbf{Q}(x, y, d; q; \kappa, \mu) = \frac{1}{1-\mu(x^2+y^2+d^2)-\kappa\hat{Q}(x, y, d; q)} \tag{5.42}$$

with a function enumerating them by anisotropic perimeter and $\triangle$-area defined similarly. Generating functions enumerating festoons by length and standard area, in which the edge variables are anisotropic or isotropic, can also be found for various weights of contact, such as osculating walk pairs, from this expression above.

The perimeter-area expressions in equations (4.36) and (4.41) are q-analogues of the perimeter generating function $\hat{P}(x, y, d)$, and that of equation (5.42) similarly related to $\mathbf{P}(x, y, d; \kappa, \mu)$. Since these perimeter-area functions are singular at $q = 1$, it is not possible to set q to 1 in the infinite sum expressions and thereby obtain the perimeter generating functions. Nonetheless, perimeter generating functions can still be obtained via the kernel method [52] (with references therein and also [53]) from intermediate results in the derivation of the infinite sums. For example, the function $\hat{P}(x, y, d)$ can be obtained by setting $q = 1$ in equation (4.35) and collecting terms.

6. First area-moment of walk pairs

Let $\widehat{TQ}(x, y, d)$ be the first area-moment generating function for parallelogram polyominoes, i.e.

$$\widehat{TQ}(x, y, d) = \sum_{p \in \mathcal{PP}} i(p) x^{a(p)} y^{b(p)} d^{c(p)}. \tag{6.43}$$

The coefficient of $x^{a(p)} y^{b(p)} d^{c(p)}$ in $\widehat{TQ}(x, y, d)$ is then the total of the standard areas of all parallelogram polyominoes with that anisotropic perimeter term: the first area-moment generating function is also known as the total-area generating function. The functions $\widehat{TQ}_\triangle(x, y, d)$, for which $\triangle$-area is counted, and $\mathbf{TQ}(x, y, d; \kappa, \mu)$ and $\mathbf{TQ}_\triangle(x, y, d; \kappa, \mu)$, which enumerate the area enclosed by contact-weighted festoons in units of standard area and $\triangle$-area respectively, are defined similarly.

In theory, all these area-moment functions for festoons (including higher moments) could be calculated as q-derivatives of the corresponding perimeter-area functions. This would be possible with a continued fraction expression for perimeter-area functions on the $\mathbb{T}$ lattice. Such an expression has, however, not yet been found, so in this section other methods are used to find the first area-moment functions for parallelogram polyominoes, and then concatenation of components is used to find first area-moment functions for festoons.

6.1. First area-moment of parallelogram polyominoes

The area of a polyomino can be found as the sum of the number of triangular cells in each column (or row) of the polyomino. If, for each column of a parallelogram polyomino, a copy of the polyomino is made with that column marked, then the term contributed by the polyomino to $\widehat{TQ}(x, y, d)$ is the same as the sum of the terms contributed by the copies so long as only the *marked area* in each copy is counted. An example is shown in figure 13.

First area-moment functions for single walks were found in [24, 54] by regrouping the heights within all walks according to the value of the height variable. Similarly, if copies of the set of parallelogram polyominoes, each with one column marked, are regrouped according to the area of the marked column, then first area-moment functions for parallelogram polyominoes can be found.

The first area-moment function $\widehat{TQ}(x, y, d)$ can be rewritten as

$$\widehat{TQ}(x, y, d) = \sum_{k \geqslant 1} k\, \widehat{TQ}_k(x, y, d) \tag{6.44}$$

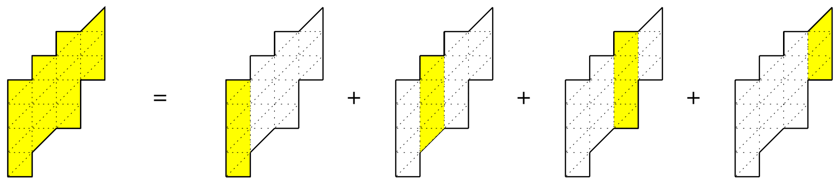


Figure 13. The area of a polyomino is the sum of the areas of its columns.

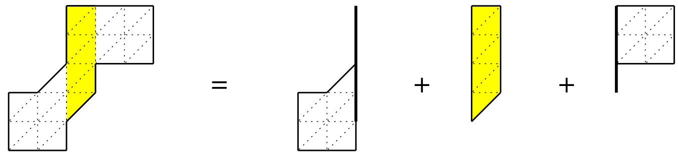


Figure 14. Decomposing a parallelogram polyomino that has a marked column.

where the sum is over the area of a marked column, and it remains to find functions $\widehat{TQ}_k(x, y, d)$ that enumerate the set of parallelogram polyominoes with a column of k cells by types of edges in the boundaries of the polyominoes.

These functions can be found by considering the possible perimeters of the parts of polyominoes that lie to the left and right of a marked column. Suppose a polyomino is split into three: the marked column, and the cells of the polyomino that lie to the left and right of the column, with the vertical edges forming the left and right heights of the column duplicated, as in figure 14. The resulting objects contain the same total marked area as the original polyomino, and if the area of the marked column is k, their combined anisotropic perimeter term is a factor of y^{2k} greater than that of the polyomino.

In section 2.2, the functions $T_n(x, y, d)$ were introduced as perimeter generating functions for the set of lattice objects remaining after the many–one compressions of the first column of parallelogram polyominoes into lines equal in height to the right height (n) of that column. It was also shown then that $T_n(x, y, d) = \lambda_-^n$, where, from equation (2.9),

$$\frac{\lambda_-(x, y, d)}{y} = \frac{1 - x^2 + y^2 - d^2 - \sqrt{(1 - x^2 - y^2 - d^2)^2 - 4(x^2y^2 + x^2d^2 + y^2d^2 + 2xyd)}}{2(y + xd)}. \tag{6.45}$$

A marked column of area k triangles and right height of r vertical edges has left height of $k - r$. In that case, the right part of a polyomino split as in figure 14 has perimeter generating function λ_-^r, and the left part, which is a similar (but rotated) object, has perimeter generating function λ_-^{k-r}.

A column on the $\mathbb{T}$ lattice with an even number of cells is bounded at top and bottom by either two horizontal edges or two diagonal edges, and a column with an odd number of cells has a horizontal edge at either top or bottom and a diagonal edge and vertical edge at the other end.

By combining these perimeter results, and dividing through by variables representing the extra vertical edges introduced by the decomposition,

$$\widehat{TQ}_k(x, y, d) = \begin{cases} (x^2 + d^2)y^{-2m}\lambda_-^{2m} & \text{if} \quad k = 2m \\ 2xdyy^{1-2m}\lambda_-^{2m-1} & \text{if} \quad k = 2m - 1. \end{cases} \tag{6.46}$$

The first area-moment generating function for parallelogram polyominoes is then

$$\widehat{TQ}(x, y, d) = \sum_{m\geqslant 1} 2m(x^2 + d^2)(\lambda_- y^{-1})^{2m} + 2(2m-1)xdy(\lambda_- y^{-1})^{2m-1}$$

$$= 2\frac{y^{-2}\lambda_-^2}{(1 - y^{-2}\lambda_-^2)^2}(x^2 + d^2 + xd(y\lambda_-^{-1} + y^{-1}\lambda_-)) \qquad (6.47)$$

which judicious but simple use of the quadratic expression for λ_- of equation (2.8) reduces to the rational expression

$$\widehat{TQ}(x, y, d) = 2\frac{(d + xy)(y + xd)(x + yd)}{(1 - x^2 + y^2 - d^2)^2 - 4(y + xd)^2}. \qquad (6.48)$$

The corresponding $\triangle$-area function, $\widehat{TQ}_\triangle(x, y, d)$, can be found by the same procedure, and is

$$\widehat{TQ}_\triangle(x, y, d) = \frac{(d + xy)(y + xd)(x + yd)}{(1 - x^2 + y^2 - d^2)^2 - 4(y + xd)^2} \qquad (6.49)$$

and since the set of parallelogram polyominoes maps to itself under reflection in a diagonal line through the origin, the total $\triangle$-area function is half of the total (standard) area function.

6.2. *First area-moment of festoons*

First area-moment generating functions for festoons do not appear to have been considered in the previous literature. In [24, 54] the first area-moment of a set of contact-weighted return single walks was expressed essentially by making copies of each walk and marking the area of one elevated walk component in each copy. The same method can be used to find the first area-moment of contact-weighted festoons from the parallelogram polyomino function $\widehat{TQ}(x, y, d)$ as

$$\mathbf{TQ}(x, y, d; \kappa, \mu) = \kappa\widehat{TQ}(x, y, d)\mathbf{P}(x, y, d; \kappa, \mu)^2 \qquad (6.50)$$

where $\mathbf{P}(x, y, d; \kappa, \mu)$ is the perimeter generating function for festoons. The corresponding $\triangle$-area function, $\mathbf{TQ}_\triangle(x, y, d; \kappa, \mu)$ is then half of this.

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